Undecidability in Relevant Logic

Søren Brinck Knudstorp
ILLC and Philosophy, University of Amsterdam

Copenhagen, August 21, 2026

SLS Symposium

Plan for the talk

- Setting and motivation
- Results
- Proof technique

Preprint available on arXiv; earlier version appears as the conference paper SBK (2024).

Prior work on (un)decidability in relevant logic

- **1953**. Mention of decision problem for $\mathsf{R}_{\to}$ (Church's (1951) weak theory of implication). [Curry and Craig]
- **1959**. $\mathsf{R}_{\to}$ is decidable. [Kripke]
- **1966**. $\mathsf{R}_{\to,\wedge}$ is decidable. [Meyer]
- **1984**. [Urquhart]
  - $\mathsf{R}^+ = \mathsf{R}_{\to,\wedge,\vee}$ is undecidable, and so are E^+ and T^+.
  - But S eluded these techniques.
  - ⇒ Prevailing expectations of S being decidable (Urquhart 2016).

Question: Is S decidable?

But first: What is S?

Defining Relevant S

Definition (language and semantics)

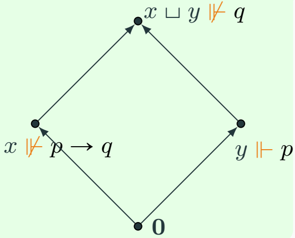
The language is given by

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

and the semantics of '$\to$' is:

$$x \Vdash \varphi \to \psi \quad \text{iff} \quad \forall y\colon\ y \Vdash \varphi \Rightarrow x \sqcup y \Vdash \psi$$

Example



Definition (frames and validity)

A frame $\mathfrak{F} = (S, \sqcup, \mathbf{0})$ is a **semilattice** $(S, \sqcup)$ with least element $\mathbf{0} \in S$; i.e.,

- *Commutative*: $x \sqcup y = y \sqcup x$,
- *Associative*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$,
- *Idempotent*: $x \sqcup x = x$.
- *Identity (least element)*: $x \sqcup \mathbf{0} = x$.

Equivalently, it is a **partial order** with all **finite joins**.
Finally, a formula φ is valid iff $\mathfrak{M}, \mathbf{0} \Vdash \varphi$ for all models $\mathfrak{M}$.

Initial goal: Prove that S is undecidable.

Broader scope, broader theme: Understanding the decidability/undecidability boundary.

The basic relevant logic B^+

Axiom schemes

1. $\varphi \to \varphi$
2. $\varphi_1 \wedge \varphi_2 \to \varphi_i$
3. $\varphi_i \to \varphi_1 \vee \varphi_2$
4. $[(\varphi \to \psi) \wedge (\varphi \to \chi)] \to [\varphi \to \psi \wedge \chi]$
5. $[(\varphi \to \chi) \wedge (\psi \to \chi)] \to [\varphi \vee \psi \to \chi]$
6. $[\varphi \wedge (\psi \vee \chi)] \to [(\varphi \wedge \psi) \vee (\varphi \wedge \chi)]$

Rules

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}\ \text{modus ponens} \qquad \frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ \text{adjunction}$$

$$\frac{\varphi \to \psi}{(\chi \to \varphi) \to (\chi \to \psi)}\ \text{prefixing} \qquad \frac{\varphi \to \psi}{(\psi \to \chi) \to (\varphi \to \chi)}\ \text{suffixing}$$

Axioms for other logics

7. $(\varphi \to \psi) \to [(\chi \to \varphi) \to (\chi \to \psi)]$ (prefixing)
8. $(\varphi \to \psi) \to [(\psi \to \chi) \to (\varphi \to \chi)]$ (suffixing)
9. $(\varphi \to \psi) \wedge (\psi \to \chi) \to (\varphi \to \chi)$ (hypothetical syllogism)
10. $\varphi \wedge (\varphi \to \psi) \to \psi$ (pseudo-modus ponens)
11. $[\varphi \to (\varphi \to \psi)] \to (\varphi \to \psi)$ (contraction)
12. $\varphi \to [(\varphi \to \psi) \to \psi]$ (assertion)

Relevant logic R:

$$\mathsf{R}^+ = \mathsf{B}^+ \oplus \{7., \ldots, 12.\} = \mathsf{B}^+ \oplus \{8., 11., 12.\}$$

Further logics:

$$\mathsf{TW}^+ := \mathsf{B}^+ \oplus \{7., 8.\} \qquad \mathsf{TWJ}^+ := \mathsf{B}^+ \oplus \{7., 8., 9.\}$$

$$\mathsf{C}^+ := \mathsf{B}^+ \oplus \{7., 8., 10.\} \qquad \mathsf{T}^+ := \mathsf{B}^+ \oplus \{7., 8., 11.\}$$

Ordered by strength:

$$\mathsf{B}^+ \subsetneq \mathsf{TW}^+ \subsetneq \mathsf{TWJ}^+ \subsetneq \mathsf{C}^+ \subsetneq \mathsf{T}^+ \subsetneq \mathsf{E}^+ \subsetneq \mathsf{R}^+ \subsetneq \mathsf{S}.$$

Routley–Meyer semantics

Routley–Meyer frames: $\mathfrak{F} = (K, R, N)$, where

$$R \subseteq K \times K \times K, \qquad N \subseteq K, \qquad x \leqslant y \text{ iff } \exists n \in N(Rnxy),$$

and:

- (preorder) $\leqslant$ is a preorder,
- (upset) $n \in N$ and $n \leqslant m$ imply $m \in N$,
- (down-down-up) $Rxyz$, $x^- \leqslant x$, $y^- \leqslant y$, $z \leqslant z^+$ imply $Rx^-y^-z^+$.

Routley–Meyer models: $\mathfrak{M} = (\mathfrak{F}, V)$ where

$$x \in V(p) \text{ and } x \leqslant y \text{ imply } y \in V(p).$$

Clauses:

$\mathfrak{M}, x \Vdash p$	iff	$x \in V(p)$,
$\mathfrak{M}, x \Vdash \varphi \wedge \psi$	iff	$\mathfrak{M}, x \Vdash \varphi$ and $\mathfrak{M}, x \Vdash \psi$,
$\mathfrak{M}, x \Vdash \varphi \vee \psi$	iff	$\mathfrak{M}, x \Vdash \varphi$ or $\mathfrak{M}, x \Vdash \psi$,
$\mathfrak{M}, x \Vdash \varphi \to \psi$	iff	$\forall y, z \in K$: $Rxyz$ and $\mathfrak{M}, y \Vdash \varphi$ imply $\mathfrak{M}, z \Vdash \psi$.

Completeness

Fact 1: B^+ is complete for all RM-frames.

Notation: Given $R \subseteq K^3$, set $x \cdot y := \{z \mid Rxyz\}$

Fact 2: Each other logic is complete for its correspondence class:

$\mathfrak{F} \Vdash (p \to q) \to [(r \to p) \to (r \to q)]$	iff	$(x \cdot y) \cdot z \subseteq x \cdot (y \cdot z)$	7.
$\mathfrak{F} \Vdash (p \to q) \to [(q \to r) \to (p \to r)]$	iff	$(x \cdot y) \cdot z \subseteq y \cdot (x \cdot z)$	8.
$\mathfrak{F} \Vdash (p \to q) \wedge (q \to r) \to (p \to r)$	iff	$x \cdot y \subseteq x \cdot (x \cdot y)$	9.
$\mathfrak{F} \Vdash p \wedge (p \to q) \to q$	iff	$x \in x \cdot x$	10.
$\mathfrak{F} \Vdash [p \to (p \to q)] \to (p \to q)$	iff	$x \cdot y \subseteq (x \cdot y) \cdot y$	11.
$\mathfrak{F} \Vdash p \to [(p \to q) \to q]$	iff	$x \cdot y \subseteq y \cdot x$	12.

Completeness

Fact 1: B^+ is complete for all RM-frames.

Notation: Given $R \subseteq K^3$, set $x \cdot y := \{z \mid Rxyz\}$, and lift to sets

$$X \cdot Y := \bigcup_{x \in X, y \in Y} x \cdot y.$$

Fact 2.5: *In particular,* TWJ^+ *is complete for the frames* $\mathfrak{F} = (K, \cdot, N)$ *satisfying*

$$\begin{aligned} &\text{(h)} & x \cdot y &\subseteq x \cdot (x \cdot y), & 7.\\ &\text{(p)} & (x \cdot y) \cdot z &\subseteq x \cdot (y \cdot z), & 8.\\ &\text{(s)} & (x \cdot y) \cdot z &\subseteq y \cdot (x \cdot z). & 9.\end{aligned}$$

How do we prove undecidability?

The Tiling Problem

- A Wang tile is a square with colors on each side.
- The tiling problem: Given a finite set of tiles $\mathcal{W}$, is it possible to tile the plane $\mathbb{Z} \times \mathbb{Z}$ so that adjacent tiles match along their shared edges.
- Introduced by Wang (1961, 1963) and proven undecidable by Berger (1966).

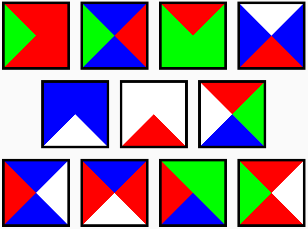


Figure 1: Wang tiles

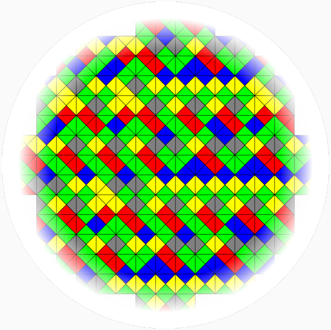


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

Given $\mathcal{W}$, construct a formula $\psi_{\mathcal{W}}$ such that:

$$
\begin{tikzcd}
 & \boxed{\mathsf{TWJ}^{+} \not\models \psi_{\mathcal{W}}} \arrow[dr, Rightarrow] & \\
\boxed{\mathsf{S} \not\models \psi_{\mathcal{W}}} \arrow[ur, Rightarrow] & & \boxed{\mathcal{W} \text{ tiles } \mathbb{Z} \times \mathbb{Z}} \arrow[ll, Rightarrow]
\end{tikzcd}
$$

Theorem

If $\mathsf{TWJ}^{+} \subseteq \mathsf{L} \subseteq \mathsf{S}$, then L is undecidable.

Theorem

$\mathsf{TWJ}^{+}, \mathsf{C}^{+}, \mathsf{T}^{+}, \mathsf{E}^{+}, \mathsf{R}^{+}, \mathsf{S}$ are all undecidable.

Two Lemmas:

1. If $\mathcal{W}$ tiles $\mathbb{Z}^2$, then $(\mathcal{P}_{\mathrm{fin}}(\mathbb{N}), \cup, \varnothing) \not\models \psi_{\mathcal{W}}$.
2. If TWJ $\not\vdash \psi_{\mathcal{W}}$, then $\mathcal{W}$ tiles arbitrarily large finite octants $\mathbb{O}(k)$.

Lemma 2: Lower Staircase

- Depict $a \cdot b \ni c$ as the arrow $a \xrightarrow[\cdot\, b]{} c$.

$$
\begin{tikzcd}
 & & & g_{3,3} \\
 & & g_{2,2} \arrow[r, "\cdot\, x_3"'] & g_{3,2} \arrow[u, "\cdot\, y_3"'] \\
 & g_{1,1} \arrow[r, "\cdot\, x_2"'] & g_{2,1} \arrow[u, "\cdot\, y_2"'] & \\
g_{0,0} \arrow[r, "\cdot\, x_1"'] & g_{1,0} \arrow[u, "\cdot\, y_1"'] & &
\end{tikzcd}
$$

Figure 3: The lower staircase for $\mathbb{O}(3)$; its construction relies on (h) and (p).

$$
\begin{aligned}
&\text{(h)} \qquad & a \cdot b &\subseteq a \cdot (a \cdot b), \\
&\text{(p)} \qquad & (a \cdot b) \cdot c &\subseteq a \cdot (b \cdot c).
\end{aligned}
$$

Lemma 2: Upper Staircase

- Depict $a \cdot b \ni c$ as $a \xrightarrow[\cdot\, b]{} c$, or as $b \xrightarrow[a\, \cdot]{} c$.

$$
\begin{tikzcd}
 & & g_{2,3} \arrow[r, dashed, "x_3\, \cdot"] & g_{3,3} \\
 & g_{1,2} \arrow[r, dashed, "x_2\, \cdot"] & g_{2,2} \arrow[r, "\cdot\, x_3"'] \arrow[u, dashed, "\cdot\, y_3"] & g_{3,2} \arrow[u, "\cdot\, y_3"'] \\
g_{0,1} \arrow[r, dashed, "x_1\, \cdot"] & g_{1,1} \arrow[r, "\cdot\, x_2"'] \arrow[u, dashed, "\cdot\, y_2"] & g_{2,1} \arrow[u, "\cdot\, y_2"'] & \\
g_{0,0} \arrow[r, "\cdot\, x_1"'] \arrow[u, dashed, "\cdot\, y_1"] & g_{1,0} \arrow[u, "\cdot\, y_1"'] & &
\end{tikzcd}
$$

Figure 4: The upper staircase for $\mathbb{O}(3)$, constructed via (s).

(s) $\qquad (a \cdot b) \cdot c \subseteq b \cdot (a \cdot c)$

Lemma 2: $\mathbb{O}(k)$ construction

$$\begin{tikzcd}
g_{0,3} \arrow[r, dotted, "x_1\,\cdot"] & g_{1,3} \arrow[r, dashed, "x_2\,\cdot"] & g_{2,3} \arrow[r, "x_3\,\cdot"] & g_{3,3} \\
g_{0,2} \arrow[u, dotted, "\cdot\, y_3"] \arrow[r, dashed, "x_1\,\cdot"] & g_{1,2} \arrow[u, dashed, "\cdot\, y_3"'] \arrow[r, "x_2\,\cdot"] & g_{2,2} \arrow[u, "\cdot\, y_3"'] \arrow[r, "\cdot\, x_3"'] & g_{3,2} \arrow[u, "\cdot\, y_3"'] \\
g_{0,1} \arrow[u, dashed, "\cdot\, y_2"] \arrow[r, "x_1\,\cdot"] & g_{1,1} \arrow[u, "\cdot\, y_2"'] \arrow[r, "\cdot\, x_2"'] & g_{2,1} \arrow[u, "\cdot\, y_2"'] & \\
g_{0,0} \arrow[u, "\cdot\, y_1"] \arrow[r, "\cdot\, x_1"'] & g_{1,0} \arrow[u, "\cdot\, y_1"'] & &
\end{tikzcd}$$

Figure 5: The octant $\mathbb{O}(3)$, constructed recursively via (p): first $\dashrightarrow$, then $\cdots\!\rightarrow$.

(p) $\qquad (a \cdot b) \cdot c \subseteq a \cdot (b \cdot c)$

Summary

Method

- Reduction from Wang Tiling
- Encoded in relational semantics

Highlights

- S is undecidable
- TWJ^+ [or (h)-(p)-(s) combined] as a source of undecidability
- $\mathsf{R}^+, \mathsf{E}^+, \mathsf{T}^+$ are undecidable (old result, new proof)

Thank you!

Related results

Let L be a bounded distributive lattice,
and $(\backslash, /, \cdot)$ the residuated family.
Consider:

$(L, \backslash)$	(this work)
$(L, \backslash, /, \neg)$	(Galatos et al. 2026)
$(L, \cdot, \neg)$	(SBK 2026)